S.-T. Yau College Student Mathematics Contests 2020

Analysis and Differential Equations

Individual-II

Please solve the following problems.

1. (a) Assume that the function $f : [a, b] \to \mathcal{R}$ with $a < b$ is differentiable and satisfies $|f^{'}(t)| \geq \beta$ for all $t \in [a, b]$ for some $\beta > 0$. Prove the following estimate

$$m\{t \in [a, b] : |f(t)| \leq \varepsilon\} \leq \frac{2\varepsilon}{\beta} \quad \text{for } \varepsilon > 0,$$

where $m\{B\}$ denotes the Lebesgue measure of set B.

(b) Assume that the function $f : [a, b] \to \mathcal{R}$ with $a < b$ is q-times continuously differentiable and satisfies $|f^{(q)}(t)| \geq \beta$ for all $t \in [a, b]$ for some positive integer q and $\beta > 0$. Prove the following estimate

$$m\{t \in [a, b] : |f(t)| \leq \varepsilon\} \leq 4\left(q!\frac{\varepsilon}{2\beta}\right)^{\frac{1}{q}} \quad \text{for } \varepsilon > 0,$$

where $m\{B\}$ denotes the Lebesgue measure of set B.

2. Suppose that $E_p(z) = (1 - z)\exp(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p})$, $p \in \mathbb{N}$. Then prove that:

$$|1 - E_p(z)| \leq |z|^{p+1}, \ \ |z| \leq 1.$$

3. Solve the following ordinary differential equation by elementary integration

$$x\frac{dy}{dx} = \sqrt{x^6 - y^2} + 3y.$$